R.I.P., SOLOW RESIDUAL

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ABSTRACT

The "Solow residual" method for measuring technological progress assumes factor income shares accurately proxy for factor marginal products. We find they never do. Using marginal products, the Solow residual always becomes zero. Technical progress, by increasing factor productivity, increases its present value, and hence selling price. The quantity of a factor may stay constant, but the dollar expenditure on it will rise if there is technological progress, due to its increased present value of expected future output. The standard production function will fully explain changes in GDP without adding Solow's "A" factor.

Introduction

Solow(1957) used factor shares as a proxy for marginal products when calculating the effect on output of changes in capital and labor. He concluded that an additional factor "A", a measure of technical progress was needed, beside simple increases in capital and labor to fully explain growth in GDP. He emphasized that this conclusion about the need for an "A" factor was totally dependent on the validity of his assumption that factor shares were accurate proxies for marginal products. This study empirically tested how accurately factor shares proxy marginal products. It found that they, at best, are very inaccurate estimates. The study also finds that when marginal products are inserted into standard Cobb Douglas production functions (instead of factor income shares). do so fully explain the GDP without requiring a separate "A" factor. Thus, Solow's residual, "A" always has a value of zero. In short, Solow's residual as a measure of technological progress, was simply wrong. Changes in the money value of capital and labor, reflecting the effect of technological progress on their present monetary value, fully incorporate technological progress into the production function. No separate "A" variable is needed.

One economic theory – consistent reason this may occur is that when the productivity of a factor of production increases, so does its present value, and this raises the factor's selling price. Hence, even if the physical quantity of the factor employed remains the same, dollar spending on the factor unit will increase, and that is how we measure capital and labor in typical production functions.

This study also finds that up until the 1980s, the marginal product of capital was far below the factor share it was paid, and labor's marginal product was far above the factor share it was paid. After 1980, that has reversed. Capital's share of income is now far less than its marginal product, and labor's share of income more than its marginal product. But profit income has risen with the increase in marginal product. That appears to be because a dollar invested oversees is yielding more profit income than a dollar invested domestically, as shown in the growth in U.S. profits from the rest of the world compared to the growth in total profits.

(We note that there is a difference between capital's share of income, and profit's share: capital's share also includes also includes interest and rental income . Unlike profits share, which has risen markedly since 1980, capital's share has remained relatively constant, due to declines in interest and rental's shares. (See Table 6 below, and also Heim 2017, Table 20.1.1.2.)

1. Theory

Productivity is a measure of the relationship between outputs (total product) and inputs i.e. factors of production (primarily labor and capital). It equals output divided by input. There are two measures of productivity: (a) labor productivity, which equals total output divided by units of labor and (b) total factor productivity, which equals total output divided by weighted average of the inputs.

A widely used production function is the Cobb-Douglas function which is as follows:

$$GDP=f(K^{\alpha} L^{\beta}) \qquad (1)$$

Where GDP is total product, K is capital, α is output elasticity of capital, L is labor and β is the output elasticity of labor. It is commonly assumed to be a constant returns to scale function, so β is commonly thought to equal (1 – α). Whether it actually does or not is an empirical question, to be answered below.

"Total factor productivity" (TFP) is a measure of productivity calculated by dividing economy-wide total production by the weighted average of inputs i.e. labor and capital. It represents growth in real output, which is in excess of the growth in inputs such as labor and capital, i.e., (A). It results from intangible factors such as technological change, education, research and development, synergies, etc. In Solow (1957), it modifies the traditional Cobb-Douglas production function in the following technologically neutral way:

$$GDP= Af(K^{\alpha} L^{\beta}) \text{ or } A = Q/f(K^{\alpha} L^{\beta}) \qquad (2)$$

Hence if changes in K and L increased Q by 100 units, and A= .05, $Q = 1.05\, f(K^{\alpha} L^{\beta}) = 105$ units

It is more useful to look at productivity increase per period of time instead of the absolute value of total factor productivity. To do this we take the logs and then the first differences of the Solow production function given above to obtain the following:

$$\Delta \text{Log } GDP = \Delta \text{Log } (A) + \alpha^{*}\Delta \text{Log}(K) + \beta^{*}\Delta \text{Log}(L) \qquad (3)$$

Which, for relatively small changes in the variables, can be closely approximated by

$$\Delta GDP/GDP = (\Delta A)/ A + \alpha * (\Delta K)/K + \beta * (\Delta L)/L \text{ or} \qquad (4)$$

$$\%\Delta GDP = \%\Delta A + \alpha^{*}\%\Delta K + \beta^{*}\%\Delta l \qquad \text{(Solow, 1957, repeated in Blanchard 2007)} \qquad (5)$$

<u>Example</u>
Consider the following production function , with arbitrarily selected α and β:

$$GDP= AK^{0.70} L^{0.45} \qquad (6)$$

If the growth in total output is 3% in a period in which capital and labor grew by 1.5% and 2%, we wish to determine the growth that is attributable to total factor productivity.

We need to isolate the increase in total product that is not explained by the increase in inputs i.e. capital and labor. Let's just punch the available data in the growth accounting equation above:

$$3\%= 0.70 \times 1.5\% + 0.45 \times 2\% + \Delta A/A \qquad (7)$$
$$\Delta A/A = 3\% -0.70 \times 1.5\% -0.45 \times 2\% = 3\%-1.95\% = 1.05\% \qquad \text{about 1/3 of total growth} \qquad (8)$$

A method for ascertaining year to year changes in productivity was needed to help answer some persistent questions about economic growth. For example, Hulten (2000) noted

> …The average GDP growth rate …(was)…1.7% from beginning of the American revolution until 1997 … how much growth is due to technology's effect on a factor, and how much to simple growth in the quantity of factor usage …Why hasn't the widely touted information revolution reversed the productivity slowdown? Robert Solow (1957) puts the proposition succinctly: "We can see the computer age everywhere but in the productivity statistics"…(Hulten 2000)

As noted earlier, Solow's 1957 formulation is given as:

$$GDP_t = A_t f(K_t,L_t) \text{ …=Hicks neutral technological progress, or} \quad (9)$$

$$\%\Delta A = \%\Delta GDP – MPK *\%\Delta K - MPL*\%\Delta AL \text{ (Solow 1957, repeated in Hulten 2000)} \quad (10)$$

Solow did not have explicit measures of MPK and MPL. But he used a long standing assumption in economics that competitive economies, capital's percentage share of income (r) accurately proxies for MPK, and labor's percentage share of income (w) is assumed to be an accurate proxy for MPL. In his own words:

> …I want to describe an elementary way of segregating variations in output…due to technical change from due to changes in the availability of capital…Naturally, every additional bit of new information has its price. In this case the price consists of…one new assumption, that factors are paid their marginal products…(Solow 1957)

In some cases they can be accurate proxies. If, the production function is a Cobb Douglas -type function, with constant returns to scale, factor income shares are equal to marginal product, and equal the elasticities of output w.r.t. changes in K or L. (Hulten 2000). However, that is an assertion built on a "if they are" assumption", a theoretical exercise. We are interested in"whether they really are, which is an empirical exercise we perform further below..

2. Sources of Data Used in Regressions:

Data used are U.S. data are taken from two sources: *The Economic Report of the President* (ERP) 2012 and 2005, Appendix Table B2: (Real GDP), and (Table B28): Nominal Compensation of Employees. Data on Real capital Stock of the U.S. (2011=100)
https://fred.stlouisfed.org/series/RKNANPUSA666NRUG

3. Methodology:

Real capital stock data (2011=100) was obtained from the Federal Reserve. It was converted from 2011 =100 base year to 2005=100 for consistency with GDP and Labor compensation data, approximated using the ratio 2009 implicit price deflator /2005 implicit price deflator because 2011 implicit price deflator data was not available at the time of writing.

Data covering the 50 year period 1961 – 2011 were analyzed. Data on percentage changes in GDP, labor and unemployment rate-modified capital were stationary (ADF test); the unmodified capital variable was nonstationary, but cointegrated with the dependent variable %Δ it was used with, so no detrending was necessary. No Hausman endogeneity was between real GDP and real K or real L, hence no instruments were needed, and therefore the model could be estimated in OLS. Newey -West standard errors were used to avoid heteroskedasticity problems. Durbin Watson autocorrelation statistics are used because most samples are small, following the small-sample recommendation of Hill, Griffith, and Lim (2011). Test were run in first differences of the data. Multicollinearity was virtually zero (r=.08) between the %Δ capital and %Δ labor variables. Near zero multicollinearity between real %ΔK and real %Δ L provided good insurance the coefficients on K and L were not distorted by collinearity between the two explanatory variables.

Solow did not use regression estimates of marginal products (α,β) in his standard production function model

$$\%\Delta GDP = \alpha*\%\Delta K + \beta*\%\Delta L.$$

Instead, Solow's production function was calculated using K and L income shares as proxies for marginal products estimates, MPK and MPL. Data on factor shares was readily available. Data on marginal products was not., but could be obtained using regression using the equation immediately above as the testable hypothesis.

But, regression is not needed, if, factor income shares can be assumed equal to their marginal products, which was Solow's assumption. Testing the assumption that factor shares = marginal products is really the main scientific objective of this study, The results will determine whether or not a standard production function $GDP = K^{\alpha} L^{\beta}$ already incorporates technological progress into the value of K and L, or whether one with a separate variable (A) measuring technological progress, such as Solow's

$$GDP = AK^{\alpha}L^{\beta}, \text{ and } \%\Delta GDP = \%\Delta A + \alpha * \%\Delta K + \beta * \%\Delta L.$$

Is needed. The standard formulation of the Cobb-Douglas model, $\%\Delta GDP = \alpha * \%\Delta K + \beta * \%\Delta L$ was tested to determine, by OLS regression, the marginal products α and β. For the 1961-2010 period these marginal products were estimated as

$\alpha = .24$ and $\beta = .77$

or in levels of the variables

$GDP = K^{.24}L^{.77}$.

The marginal product estimates varied considerably with the factor incomes shares for the same period. For reasons described below, we feel it was differences in technological progress in different periods that affected a particular factor's productivity.

Two tests were undertaken to determine if Solow's technological progress variable (A) had a value greater than zero:

1. A second regression was then run to determine if there is any significant difference in the level of output the first regression predicts and the $GDP = \mu\ (.24\%\Delta K + .77\%\Delta L)$. A statistically significant estimate of $\mu > 1.00$ was taken as indicating the technological progress was not accounted for by the standard production function.

2. A third, even simpler, regression test was undertaken to see if the results of the second regression could be verified. In this test, a constant was added to the standard formula to obtain to obtain a value for the %ΔA, i.e.:

   $\%\Delta GDP = \%\Delta A + \alpha * \%\Delta K + \beta * \%\Delta L$

The whole range of these tests are also repeated using NNP instead of GDP, and using K reduced by the unemployment rate as a way of distinguishing total capital available from capital in use. Both are adjustments recommended by Solow, the latter of which was actually used in Solow (1957).

We note that (α, β) may be thought of interchangeably as marginal products of %ΔK, %ΔL or as the elasticity with which a given %Δ K, or %Δ L brings about a %ΔGDP. In the findings below, they are sometimes referred to as one, sometimes as the other depending on context.

5. Findings

OLS Regression was used to obtain estimates of output elasticities (α, β) for capital (K) and labor (L) for the Cobb-Douglas production function. Estimates and their confidence levels, represented by t-statistics, are presented in Table 1 below.

Table 1
Empirical vs. Solow Factor Share Estimates of α, β

Period Tested		α %ΔK (t stat)	β %ΔL (t stat)	R^2_{Adj}	Durbin Watson
1961 -2010	%ΔGDP =	.244 %ΔK (2.7)	.774 %ΔL (10.0)	78.2%	2.0
1961 -1982	%ΔGDP =	.12 %ΔK + (0.9)	.83 %ΔL (7.4)	78.7%	2.1
1983 -2000	%ΔGDP =	.56 %ΔK (2.3)	.59 %ΔL (3.5)	54.3%	1.7
1983 -2010:	%ΔGDP =	.48 %ΔK (2.7)	.66 %ΔL (5.5)	77.1%	1.7
Solow Factor Shares: 1909-1948	%ΔGDP =	.32 %ΔK	.68 %ΔL	NA	N.A.

By comparison, Solow's model used factor share proxies, not empirical estimates of the values of marginal products α and β. Those shares were α = .32 and β = .68 (Solow 1957). We show below that use of any marginal product estimates other than those empirically obtained, typically leads to a major overestimate of the value of (A), the variable denoting the growth in GDP in excess of the growth in capital and labor.

Next, to find the Solow Residual (A), we estimated (α, β) it from equation (10) above:

Solow's 1957 formulation is given as:

$GDP_t = A_t f(K_t,L_t)$ …Using Hicks neutral technological progress 9 (Repeated)

Which, after logging and taking time derivatives of the data becomes

or $\%\Delta A = \%\Delta GDP – MPK *\%\Delta K - MPL*\%\Delta L$ 10 (Repeated)

or equivalently,

$$\%\Delta A = \%\Delta GDP – (MPK *\%\Delta K + MPL*\%\Delta L) \quad (11)$$

Dividing through by (MPK *%ΔK + MPL*%ΔAL) and rearranging gives

$$1+\%\Delta A / \textbf{(}MPK*\%\Delta K + MPL*\%\Delta L\textbf{)} = \%\Delta GDP / (MPK *\%\Delta K + MPL*\%\Delta L) \quad (12)$$

Hence, for example, if

4% = Total %ΔGDP, and
3% = % Δ in GDP generated by (α,β) changes in K and L,
1% =% Δ in GDP generated by %ΔA

Then, rearranging the numbers a bit, we find

(1 + .33) = .04/.03 = 1. 33 since %ΔA / (MPK*%ΔK + MPL*%ΔL) = 1/3 =.33…by assumption

Note: If all % changes in the GDP are fully accounted for by % changes in K and L, then 1+(%ΔA//(MPK *%ΔK + MPL*%ΔAL) = 1.00, which the most common finding, as we show in Table 2 below.

Table 2
Estimates of Solow Residual

Period Tested		Ratio of: %ΔGDP/ (MPK*%ΔK+MPL*%ΔL)	R^2_{Adj}	Durbin Watson
Using Estimate MPK, MPL:				
1961 -2010	%ΔGDP =	1.00 (t=25.9)	78.6%	2.0
1961 -1982	%ΔGDP =	1.00 (t=17.3)	81.3%	2.1
1983 -2000	%ΔGDP =	1.00 (t=13.0)	57.3%	1.7
1983 -2010:	%ΔGDP =	1.00 (t=16.7)	78.0%	1.7
Using Solow's Factor Shares as Proxies for MPK, MPL:				
1961-2010	%ΔGDP =	1.04 (t=25.7)	78.2%	2.1

If the effects of technological progress on GDP is reflected completely in changes in the money value of K and L, i.e., GDP = K^{α},L^{β}, actual %Δ GDP should not be greater than the % change due to changes in K, and L alone, .e.g., in Table 1, .244 %Δ ΔK +.774 %Δ ΔL for 1961-2010. This implies the Table 2 ratio should be 1.00, i.e., the % ΔA = 0 in the expression 1 + %ΔA/(MPK*%ΔK + MPL*%ΔAL).

1.00 is, in fact, the ratio we obtained for all test periods. This finding indicates a "Solow Residual" variable is not needed to explain technological progress, as the effects of technological progress are already accounted for in the standard Cobb-Douglas production function as estimated in Table 1. The result also indicates the coefficient on the "%ΔA" variable, when included in a production function equation like (10) above, should be zero. As we show below, that is exactly what we get.

Why might this be? Microeconomics tells us that in an economy with competitive output markets, firms increase their use of a factor until its marginal cost equals its marginal value (=Price). Marginal cost includes a provision for normal profit per unit. If the marginal cost of a factor does not change, despite an increase in productivity increasing the factor's value, the supply curve will shift to the right, and more units of the factor will be produced until MC again equals MR, the profit maximizing level.

But if suppliers of factors of production can set the market price for their factor, things are different. It means they can adjust the profit component of MC to reflect any increase in factor productivity due to Solow's technologic progress, rather than simply increase the number of (now higher productivity) units of the factor sold.

From the factor supplier's point of view, this is probably the preferred alternative. it is reasonable to believe competitive factors should result in a factor being priced in accordance with its present value, or in the case of labor, the rental value of the unit. If a unit of the factor will now produce 50% more product, due to technological progress, why would the supplier not charge 50% more for the factor? Hence, when the marginal productivity of a machine (or worker) increases 50%, if producers have control over the prices of the factor they produce, we should see the factor also raise its price, ideally 50%, but less if producers have less than full control of pricing. An increase in spending on a factor can be *assumed* to result from an increase in physical units of capital or labor sold (since we are measuring K and L in terms of their dollar shares of national income). But in fact, it may be an increase in the per unit price of K or L sold, due to technological progress, not an increase in the number of physical units sold, that is driving up spending on the factor. E.g. the present value of a 10 year life machine whose output has now doubled due to technological progress, is doubled. In short, when calculating the production function in the standard way, technological progress is fully accounted for by changes in the total spending on K and L.

$$PV_0 = \Sigma_{0\text{-}9}(\text{ return}_{.i} /(1+\text{discount rate})^{I}) \quad \text{(where i=0-9periods in the future)} \qquad (13)$$

Conclude:

If we have estimated the production function correctly, and increases in technology are priced into the selling price of of the factors of production, the coefficient reflecting GDP as measured by the production function, should just equal the "actual" GDP. Our ratios in Table 2 should be 1.00. This is exactly what our results show. If the typical annual growth in the GDP is 3% the ratio indicates technological progress only accounts for 0.0 % of the 3% growth, the changes in K and L captured within the production function, whether they be quantitative or qualitative, account for all of the 3% growth.

Why then do historical estimates of "A" all suggest that it is significantly greater than zero? Historical results, starting with Solow's, show a technological progress variable (A) with a value greater than zero, appear to result from erroneously assuming, that factor shares received by capital and labor are equal to their elasticities in the production function, and using factor shares of income as a proxy for elasticities. Empirical results show they are not good proxies.

Standard production function regressions should not show output growing by changes total spending on the Cobb-Douglas's inputs, plus some extra amount (A) due to technological advances. The empirical evidence indicates the "extra" amount is already accounted for In any increase in capital and labor spending. Hence, our regressions in Table 2 showed a 1.00 coefficient (with extremely high t statistics, typically ~ t=16-25) on the Cobb-Douglas's ability to calculate output equal to the actual GDP. Table 3 below shows the difference in estimated value of the technological progress variable (1+%ΔA/(α %ΔK + β %ΔL) using both regression estimates of α and β, and also using capital and labor income shares.

Table 3
National Income Shares As Proxies for MPK, MPL When Estimating the Solow Residual
(Using GDP, Total K Stock)

Period	MPK	MPL	A's R^2 *	1+A'=1+%ΔA / ƒ (%ΔK,%ΔL)	Method
Solow's Model Using Factor Share Proxies for MPK, MPL in %ΔGDP = MPK%ΔK,+MPL%ΔL					
1909-1948	.32	.68	N.A.	1+A'=1.04	Factor Shares Estimate (That Assume Factor Shares=MPK, MPL)
Using Empirical Estimates of MPK, MPL in %ΔGDP = MPK%ΔK,+MPL%ΔL					
1961-2010	244	.774	(A's R^2_{Adj}=.786)	1+A'=1.00	Using MPL,MPK in %Δ GDP= MPK*%Δ K +MPL*%Δ L (Not Factor Share Proxies))
	.355	.645	(A'sR^2_{Adj}=.778)	1+A'=1.05	Using Factor Income Shares As Proxies For Estimated MPK, MPL In

					%Δ GDP= MPK*%Δ K +MPL*%Δ L)
1961-1980	.159	.801	(A’sR^2_{Adj}=.810)	1+A’=1.00	Using Marginal Products
	.356	.644	(A’sR^2_{Adj}=.786)	1+A’=1.00	Using Factor Shares
1980-2000	.149	.871	(A’sR^2_{Adj}=.706)	1+A’=1.00	Using Marginal Products
	.347	.653	(A’sR^2_{Adj}=.687)	1+A’=1.08	Using Factor Shares
1990-2010	.434	.647	(A’sR^2_{Adj}=.822)	1+A’=1.00	Using Marginal Products
	.358	. 642	(A’sR^2_{Adj}=.821)	1+A’=1.07	Using Factor Shares
Four Sample Averages:			(A’sR^2_{Adj}=.778);	1+A’ = 1.00	Using Marginal Products
Four Sample Averages:			(A’sR^2_{Adj}=.768);	1+A’ = 1.05	Using Factor Shares

*A’sR^2_{Adj} = Adjusted R^2 for equation estimating (1+ (%ΔA/((α%ΔK,+β%ΔL))

Table 3 results indicate that factor shares are generally widely disparate from marginal productivity estimates. To the extent they differ from econometrically obtained marginal productivity estimates for α and β, they lead to incorrect estimates of total factor productivity (TFP): generally by indicating (1+ (%ΔA/((α%ΔK,+β%ΔL)) is greater than 1.00; only in one of the samples (1961-80)were they equal to one. 1.00. Use of the marginal products increased explained variance (R^2) by 1.0% on average for the four samples.

In all cases, using empirically estimated marginal products for α and β clearly indicated there was no additional effect of technological progress (A) on the GDP other than that given by subtracting from GDP the part of the GDP calculated from the standard production function formula, using the money value of aggregate capital and aggregate employment to define those variables.

The finding of earlier studies that technological progress could be measured by growth in GDP minus growth in output due to the in the traditional production function, seems to substantially different from those of this study using the approach shown in Table 3. The divergence seems to have resulted from mistakenly assuming the economy was sufficiently competitive to allow the substitution of factor shares as good proxies for marginal products when estimating the production function.

When properly calculated using marginal products instead of factor shares, the Solow residual, defined as a growth in GDP over and above that measured by the production function, is found to be zero. Solow, by his own admission, banked the validity of his findings on the assumption factor shares were a good proxy for marginal products. The assumption appears to have been wrong.

Modifications to the Standard Model

Solow (1957) argued that

1. using Net National Product (NNP) and

2. using a variant of the K variable that reduces the stock of capital by the labor unemployment rate (to differentiate total capital available, from the level of capital actually used)

would provide a more accurate measures of how technological progress would affect the GDP.

To test these two ideas, results for all four time periods tested in Table 3 above have been reestimated. We test the same models as above, except using NDP instead of GDP, and by using total capital reduced by the unemployment rate, or both. Results are shown in Table 4 below.

Table 4
National Income Shares As Proxies for MPK, MPL When Estimating the Solow Residual
(Using NDP and Unemployment Rate Reduced K Modifications)

Period	MPK	MPL	A's R^2 *	1+A'=1+(%ΔA/ ƒ(%ΔK,%ΔL)	Method
NDP Modification (Only)					
1961-2010	.116	.883	(A's R^2_{Adj}=.749)	1+A'=1.00	Using marginal products
	.354	.646	(A's R^2_{Adj}=.778)	1+A'=1.06	Using Factor Shares Shares As Proxies For MPK, MPL In NDP= A(MPK*K +MPL*L)
1961-1980	.0195	.920	(A's R^2_{Adj}=.796)	1+A'=1.00	Using Marginal Products
	.356	.644	(A's R^2_{Adj}=.743)	1+A'=1.01	Using Factor Shares
1980-2000	.1412	.9049	(A's R^2_{Adj}=.626)	1+A'=1.00	Using Marginal Products
	.347	.653	(A'sR^2_{Adj}=.609)	1+A'=1.12	Using Factor Shares
1990-2010	.3477	.7050	(A's R^2_{Adj}=.7675)	1+A'=1.00	Using Marginal Products
	.358	.642	(A's R^2_{Adj}=.7669)	1+A'=1.06	Using Factor Shares
Four Sample Averages:			(A'sR^2_{Adj}=.735);	1+A'=1.00	Using Marginal Products
Four Sample Averages:			(A's R^2_{Adj}=.710);	1+A'=1.06	Using Factor Shares
NDP and Unemployment Rate Reducion of K					
1961-2010	.354	. 702	(A's R^2_{Adj}=.7612)	1+A'=1.00	Using Marginal Products
	.354	. 646	(A's R^2_{Adj}=.7611)	1+A'=1.06	Using Factor Shares As Proxies For MPK, MPL In NDP= A(MPK*(i-Umem%)*K +MPL*L)
1961-1980	.053	.894	(A's R^2_{Adj}=.80)	1+A'=1.00	Using marginal products
	.356	.644	(A's R^2_{Adj}=.78)	1+A'=1.02	Using Factor Shares
1980-2000	.641	.536	(A's R^2_{Adj}=.714)	1+A'=1.00	Using marginal products
	.347	.653	(A's R^2_{Adj}=.707)	1+A'=1.08	Using Factor Shares
1990-2010	.654	.471	(A's R^2_{Adj}=.78)	1+A'=1.00	Using marginal products
	.358	.642	(A's R^2_{Adj}=.77)	1+A'=1.06	Using Factor Shares
Four Sample Averages:			(A's R^2_{Adj}=.764);	1+A' = 1.00	Using Marginal Products
Four Sample Averages:			(A's R^2_{Adj}=.755);	1+A' = 1.06	Using Factor Shares
GDP and Unemployment Rate Reduced K Modification					
1961-2010	.469	. 598	(A's R^2_{Adj}=.795)	1+A'=1.00	Using marginal products
	.354	. 646	(A's R^2_{Adj}=.793)	1+A'=1.04	Using Simulated Factor Income

					Shares As Proxies For Est. MPK, MPL In GDP =A(MPK*1-Umem%)*K +MPL*L)
1961-1980	.185	.779	(A's R^2_{Adj}=.803)	1+A'=1.000	Using Marginal Products
	.356	.644	(A's R^2_{Adj}=.796)	1+A'=1.004	Using Factor Shares
1980-2000	.8224	.389	(A's R^2_{Adj}=.7889)	1+A'=1.00	Using marginal products
	.347	.653	(A's R^2_{Adj}=.762)	1+A'=1.09	Using Factor Shares
1990-2010	.7578	..3953	(A's R^2_{Adj}=.830)	1+A'=1.00	Using marginal products
	.358	.642	(A's R^2_{Adj}=.802)	1+A'=1.05	Using Factor Shares
Four Sample Averages:			(A's R^2_{Adj}=.804);	1+A' = 1.00	Using Marginal Products
Four Sample Averages:			(A's R^2_{Adj}=.789);	1+A' = 1.04	Using Factor Shares

*AR^2_{Adj} = Adjusted R^2 for technical change Variable (A).

Conclude: None of the 12 tests in Table 4 using marginal products, as production theory indicates is appropriate, show the Solow residual to have any value but zero. Nor did any of the four Table 3 tests.

By comparison, all 12 of the Table 4 tests, and three of the four Table 3 tests showed the Solow residual increasing at between 4/10ths of 1% and 12% a year using factor shares as a proxy for marginal products.

The residual only shows a positive value, indicating GDP growth in excess of what the standard production function shows in periods when factor shares not equal to marginal products are used (which appears to be virtually always)

The difference in results appears due to assuming factor shares are a good proxy for marginal products, when in fact they rarely are.

The real value of "A" as measured using the Solow method is zero, since its effects are already totally captured by the growth in K and L each period. This appears to be because changes in the dollar measures of capital, or employment used each year in the production function, pick up the present dollar value of both quantitative and qualitative changes to capital and labor, as we discuss in the next section,

The Theory of Why %ΔA=0 (or 1+A'=1.00)

The supply curve in competitive markets is the firm's MC curve. Note in Table 5 below the same rightward (or downward) shift in the MC curve from (S1) to (S2) could occur for three different reasons:

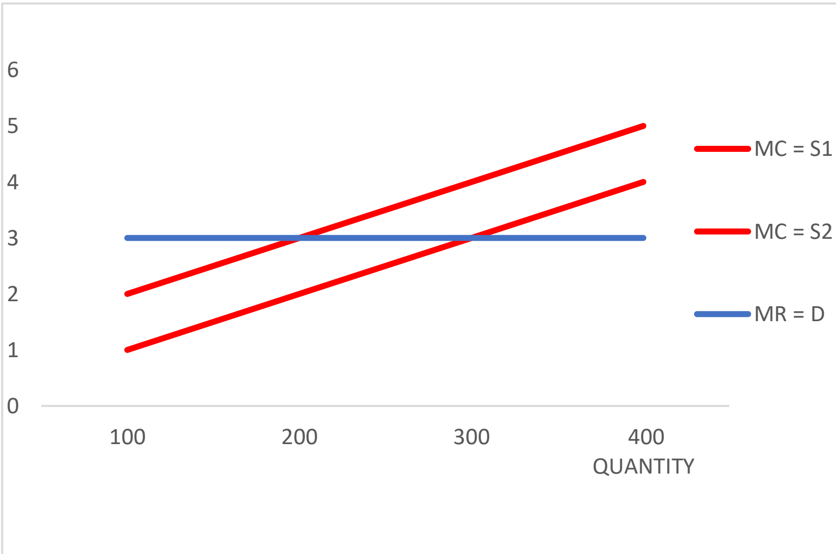
1. Quantitative Increase in factor usage, increasing output, using same technology. Same Eq. Price
2. Quantitative Increase in factor usage by new entrants, using same technology. Same Eq. Price
3. Increased productivity of factors due to new technology, output increases factor usage and Eq. price. unchanged

All three types of changes in factor input yield the same result: a rightward (or downward) shift in the supply curve from S1 to S2. Price remains the same, quantity produced increases (Graph 1). Total expenditure on the factor (e.g., capital) increases. Since any of the three changes leads to increased spending on the factor, it is impossible to tell from the output price and quantity data on changes on factor spending which has occurred: a qualitative or quantitative increase in the factor of production used. The change in spending on the factor can be caused by any of them, including the effects of technological progress. Hence, there should be no effect on output over and above what is shown in the production function. However, if the productivity of the factor increases, its marginal product changes, leading to a change in output elasticities, i.e., a change in MPK or MPL in

$$\%\Delta GDP = MPK*\%\Delta K + MPL*\%\Delta L \tag{14}$$

Which again leaves the standard production function completely able to explain the effects of technological growth on output. No additional "A" factor is needed.   Which explains why in tests using marginal products, "A" always is found to be A=0.

Graph 1
Horizontal Market Demand Curve



Graph 1 assumes the demand curve paced by producers is horizonal, i.e., producers have no control over price. In Graph 2 below, we assume the producers are large enough so that changes in output due to changes in factor usage affect price. The demand curve is down sloping, and marginal revenue declines as production increases. The producer still maximizes profit by producing up to the point where MR=MC and obtains the price buyers are willing to pay for that much product.

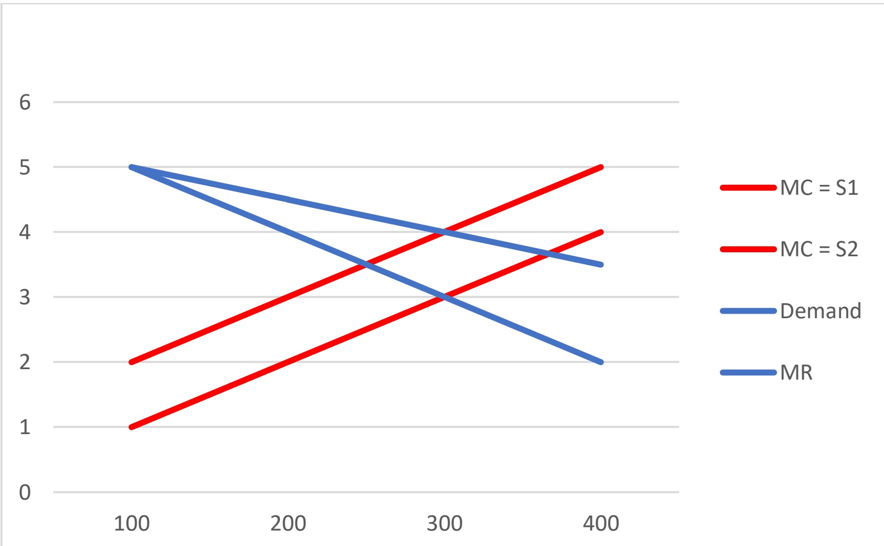
Note in Graph 2 below the same rightward shift in the S1 supply curve to S2 could occur for three reasons:

1. Quantitative Increase in factor usage, by same firms, using same technology
2. Quantitative Increase in factor usage by new entrants, using same technology
3. Quantitative increase, decline, or constant levels of factor usage, due to technological Increase in productivity of factor. A highly elastic demand curve is likely to increase demand so much that even with higher productivity levels, increase factor usage may be required to produce the necessary supply. Highly inelastic demand curves indicate growth in demand is small, smaller than the increase in factor productivity, hence demand for the factor will decline. In some cases, the increase in demand and the increase in productivity may be the same, suggesting spending on the factor will neither increase or decline.

All three types of changes in output yield the same result: a rightward (or downward) shift in supply from S1 to S2. Price drops but quantity produced increases.

It is impossible to tell from data on changes in factor spending which has occurred. The change in spending on the factor can occur for any one of the three reasons cited above, including the effects of technological progress. If it is technological progress, the full effect of the technological change is captured in our measure of total spending on the factor experiencing technological growth. As a result, the value of the growth in output not accounted for by changes in K,L in the production function (i.e., A) should be zero. That is, when measured correctly using the Solow methodology, A=0, or (1+A) = 1.00.

Graph 2
Declining Market Demand Curve



But when output from the production function is incorrectly measured, due to use of erroneous estimates of factor marginal products resulting from using factor shares as a proxy, invariably the output produced by the "real" production function will differ from that predicted by the erroneous one. The differences can be positive or negative, large or small, but they will almost always be there, give "A" a non-zero value, as we saw in our examples above.

In one final case we note that if producers can control prices, and decide to maintain pre -technological change levels of production, spending on the factor whose productivity has increased will decline proportionately. S1 will not fall to S2 at all. It will stay at S1, reflecting the fact that though costs per unit of one factor has been reduced, costs of another (desired profit per unit) will have increased the same amount.

6. Additional Tests of Solow Hypothesis

There is an even more straightforward way of testing the need for a separate technological progress variable in the production function. Recall that Solow's 1957 formulation, described in equations (9) and (10) above, was given as:

$GDP_t = A_t f(K_t, L_t)$ …Using Hicks neutral technological progress 9 (Repeated))

Which after conversion to logs and first differencing can be written

$\%\Delta A = \%\Delta GDP - MPK * \%\Delta K - MPL * \%\Delta L$ (Hulten 2000) 10 (Repeated)

Rearranging terms a bit, this provides an easy way to use regression to estimate empirically the value of $\%\Delta A$:

$$\%\Delta GDP = \%\Delta A + MPK * \%\Delta K + MPL * \%\Delta L \quad (15)$$

Where the value of %ΔA is obtained by adding a constant term, essentially a dummy variable, to the standard production function. Test results in Table 5 below were obtained testing this model, but using as the capital variable K*(1-unem rate), since in earlier tests form of the variable explained more variance.

If are test results cited in Tables 3-4 above are accurate, we should find the find the value of %ΔA zero or not significantly different from zero. Table 5 below shows the results obtained for four different tests of different parts of the 1960-2010 period using OLS (because of no endogeneity),

Table 5
Regression Estimates of %ΔA

Period Tested	Average %ΔA	t-stat	Model R^2_{Adj}	Average %ΔGDP	Average $\%\Delta K_{1\text{-}unem}$	Average %ΔL
1961-2010	0.6%	(1.65)	78.6%	3.0%	2.8%	2.9%
1961-2008	0.3%	(0.8)	75.4%	3.1	2.9	3.1
1961-1980	-0.0%	(-0.1)	78.8%	3.4	3.0	3.4
1981-2000	0.7%	(0.1)	65.2%	3.2	2.6	3.1
1990-2010	1.0%	(1.0)	81.4%	2.4	2.3	2.2
1990-2008	-0.2%	(-0.2)	71.5%	2.7	2.4	2.6
1992-2008	0.2%	(0.2)	65.8%	2.9	2.4	2.8
2000-2010	1.1%	(1.0)	78.8%	1.8	2.1	1.4
2000-2008	-0.8%	(-0.4)	55.8%	2.2	2.4	2.0

In all 8 periods tested, the "A" variable value was statistically insignificant from zero, confirming the results found in Tables 3-4 above.

Even using the more lenient, one-tail t-test criteria,(95% critical value = 1.80 for n=50), the t=1.65 result obtained for the1961-2010 sample indicates the estimate of %ΔA obtained for that same is not significantly different than zero. Out of dozens of different sized sample periods tested (only 8 of which are shown above), only samples that included 1960s or 1970s data with samples large enough to include the 2009 and 2010 observations showed the separate variable for technological growth factor significant or near significant when testing the production function. When rerun but excluding the 2009 and 2010 data, none but one showed the technological progress variable %ΔA to be statistically significant.

In short, all the 40 subsamples tested ending with 2008 data (with one exception) reject the separate technological progress variable hypothesis. The same result was obtained adding the 2009-2010 data to samples starting with 1980 or later data. But for samples including the 1960s and 1970s data, all but one were insignificant until we added the 2009-2010 data. It is not clear why. When samples starting with the 1980s or 1990s data were used, adding the 2009 and 2010 again left the separate technological progress variable statistically insignificant.
But we do feel that despite this anomaly the overall results firmly support the hypothesis that when properly calculated, the value of the Solow residual A is zero.

Conclusion

The overwhelming preponderance of the evidence indicates that when properly added to a production function model, i.e., one using actual marginal product estimates, not factor share proxies, the results show the Solow residual, a measure of technological progress will be zero. This is because the effects of technological progress on K and L are already picked up dollar value measures of capital and labor used, through its effect on the present value (sale price) of the factors of production.

If we want to find the actual value of TFP, we are going to have to find a different method. Though technological progress exists, the Solow residual does not remotely accurately measure it.

7. Post script: A Note on Inequality of Marginal Productivities and Income Shares

In competitive factor markets, each factor is paid its marginal product. Hence in the aggregate, as well as individually, factors shares of income are equal to factor marginal products. Solow relied heavily on the assumption that market were sufficiently competitive so that he could use factor shares as reasonably accurate proxy for marginal products when calculating how much output was produced by a given amount of capital and labor. Instead of using marginal products in estimating

$$\%\Delta GDP = \%\Delta A + MPK*\%\Delta K + MPL*\%\Delta L \quad \text{(15 from above)}$$

he used

$$\%\Delta GDP = \%\Delta A + ISK*\%\Delta K + ISL*\%\Delta L \quad (16)$$

where ISK, ISL = income shares of capital and labor.

We wish here to compare trends in marginal products and in factor shares over the 1960 – 2010 period, using the data from Tables 3 and 4 above, with a few additional time periods added. Results are shown in Table 6 below for two models examined above that best explain how K and L are related to GDP over the 1960-2010 period. They are the

$\%\Delta GDP = (MPK*\%\Delta K + MPL*\%\Delta L)$ compared to the $\%\Delta GDP = (ISK*\%\Delta K + ISL*\%\Delta L)$ model

and the

$\%\Delta GDP = (MPK*(1\text{-}Unem\%)*\%\Delta K + MPL*\%\Delta L)$ compared to the $\%\Delta GDP = (ISK*(1\text{-}Unem\%)*\%\Delta K + ISL*\%\Delta L)$ model

Where the (1-Unem%) modifier was used to distinguish capital stock actually used from total capital stock available. This model typically generated higher R^2s than the model using only total capital available (K).

Table 6
Comparing Same-Period Marginal Productivities and Income Shares

	GDP = ƒ(K,L) Model				GDP = ƒ((1-Unem%)*K,L) Model			
Period	MPK	ISK	MPL	ISL	MPK	ISK	MPL	ISL
1961-1980	.159	.356	.801	.644	.185	.356	.779	.644
1961-1990	.148	.351	.845	.649	.324	.351	.707	.649
1980-2000	.149	.347	.871	.653	.822	.389	.389	.653
1990-2000	.457	.354	.603	.646	..689	.455	.454	.646
1990-2010	.434	.358	.647	.642	.758	.356	.395	.642
1961-2010:	.244	.354	.774	.646	.469	.354	.598	.646

Table 6 clearly indicates that factor marginal products and factor shares were never equal during the 1960-2010 period, and that in the 1960 to 1980 or 1990 period, factor shares for capital far exceed its marginal product (which also means factor shares for labor were far below its marginal product).

From 1980 or 1990 to 2010, we have just the opposite. Capital's marginal product far exceeds its factor share, and labor's marginal product is far less than its factor share.

Use of Solow's "actually used" definition of capital is probably the more economically sensible of the two models tested in Table 6, and it indicates the transformation begins around 1980.

This is 100% consistent with the Heim (2017, Table 20.4.3.3) finding that, controlling for other factors, the shift in factor shares since 1980 can totally be accounted for by growth in profit income due to overseas investment by U.S. businesses. The same study shows that while total labor income has risen since the1980s (Heim 2017, Table 20.1.1.1), total profit income has grown so much faster as to result in a marked increase in profit's share of national income. It is also consistent with the Heim 2017. Cptr. 20, Table 4.1.3 finding of declining labor productivity since the 1980s, which one would expect would be related to declining labor factor shares.

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